SURAT KETERANGAN

Nomor : NJ-T06/0351/A.03/LP3M/02.2021

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Judul : *Terwilliger Algebras Of Some Group Association Schemes*

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Identitas : Mathematical Journal of Okayama University, Vol. LXI, No.1, Januari 2019, E-ISSN : 0030-1566, Okayama University

No. Pemeriksaan :012085

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TERWILLIGER ALGEBRAS OF SOME GROUP ASSOCIATION SCHEMES

TERWILLIGER ALGEBRAS OF SOME GROUP ASSOCIATION SCHEMES

Nur HAMID AND MANABU OURA

ABSTRACT. The Terwilliger algebra plays an important role in the the- ory of association schemes. The present paper gives the explicit struc- tures of the Terwilliger algebras of the group association schemes of the finite groups $PSL(2,7)$, A_6, and S_6.

1. INTRODUCTION

Association schemes enable us to study combinatorial problems in a unified way. We refer to [2, 6] for the foundations of association schemes. In a series of papers [10, 11, 12], Terwilliger introduced a new method, the so-called Terwilliger algebra, to investigate the commutative association schemes. Since then there have been many investigations on Terwilliger algebras (*cf.* [8, 7]). It is very important to know the explicit structure of the Terwilliger algebra. The cases of the group association schemes of S_5 and A_5 were studied in [1] along the line of the work [3]. In the present paper we determine the structures of the Terwilliger algebras of the group association schemes of the finite groups $PSL(2, 7)$, A_6, and S_6.

The computations were done with Magma [5] and SageMath [9].

2. PRELIMINARIES

We begin with the definition of a group association scheme.

Definition 1. Let G be a finite group and $C_0 = e$, C_1, . . . , C_d the conjugacy classes of G, where e is the identity of G. Define the relations $R_i (i = 0, 1, . . . , d)$ on G by

$$(x, y) \in R_i \iff yx^{-1} \in C_i.$$

Then $X(G) = (G, \{R_i\}_{i \leq d})$ forms a commutative association scheme of class d called the *group association scheme of G*.

We associate the matrix A_i of the relation R_i as

$$(A_i)_{x,y} := \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{k=0}^{d} A^{k}_{ij} A = k_{ij}$$

and $A_0, \ldots, A_d$ generate the so-called *Bose-Mesner algebra* A. The intersection numbers p^{k}_{ij} of the group association scheme $X(G)$ are given by

$$|\{(x, y) \in C_i \times C_j | xy = z, z \in C_k\}|.$$

The algebra A has a second basis $E_0, \ldots, E_d$ of primitive idempotents, and

$$E_i \circ E_j = \frac{1}{|G|} q^{k}_{ij} E_k,$$

where $\circ$ denotes Hadamard (entry-wise) multiplication. For each $i = 0, \ldots, d$, let E_i^* and A_i^* be the diagonal matrices of size $|G| \times |G|$ which are defined as follows.

$$(E_i^*)_{x,x} := \begin{cases} 1, & \text{if } x \in C_i \\ 0, & \text{if } x \notin C_i \end{cases} \quad (x \in G),$$

$$(A_i^*)_{x,x} := |G|(E_i)_{e,x} \quad (x \in G).$$

Then $E_0^*, \ldots, E_d^*$ form a basis for the *dual Bose-Mesner algebra* A*. The intersection numbers provide information for our structural results to follow. We refer to the following relations [10].

$$E_i^* A_j E_k^* = 0 \quad \Leftrightarrow \quad p^{k}_{ij} = 0 \quad (0 \le i, j, k \le d),$$
$$E_i A_j^* E_k = 0 \quad \Leftrightarrow \quad q^{k}_{ij} = 0 \quad (0 \le i, j, k \le d).$$

We need to fix the ordering of the conjugacy classes. The following table gives the representatives and the orders of conjugacy classes.

(1) $PSL(2, 7)$

	C_0	C_1	C_2
rep.	(1)	(357)(468)	(2354786)
$|C_i|$:	1	56	24
	C_3	C_4	C_5
rep.	(2465837)	(12)(34)(58)(67)	(1235)(4876)
$|C_i|$:	24	21	42

(2) A_6

	C_0	C_1	C_2	C_3	C_4
rep.	(1)	(12)(34)	(123)	(123)(456)	(1234)(56)
$|C_i|$	1	45	40	40	90
	C_5	C_6			
rep.	(12345)	(12346)			
$|C_i|$	72	72			

(3) S_6

	C_0	C_1	C_2	C_3
rep.	(1)	(12)	(12)(34)	(12)(34)(56)
$|C_i|$	1	15	45	15
	C_4	C_5	C_6	C_7
rep.	(123)	(123)(45)	(123)(456)	(1234)
	40	120	40	90
	C_8	C_9	C_{10}	
	(1234)(56)	(12345)	(123456)	
	90	144	120	

Finally we give the definition of the Terwilliger algebra of the group association scheme. We shall denote by M_k the ring of $k \times k$ matrices over the complex number $\mathbf{C}$.

Definition 2. Let G be a finite group. The *Terwilliger algebra* $T(G)$ of the group association scheme $X(G)$ is $M_{|G|}$ sub-algebra of M_G generated by A and A^*.

Since $T(G)$ is closed under the conjugate-transpose, $T(G)$ is semi-simple. In the next section, we investigate the Terwilliger algebras of the group association schemes of $PSL(2, 7)$, A_6 and S_6.

3. RESULTS

In [1], Balmaceda and Oura gave the structures of the Terwilliger algebra of the group association schemes of S_5 and A_5. Following their method, we determine the Terwilliger algebras for the cases $PSL(2, 7)$, A_6, and S_6.

Theorem 3.1. *The dimensions of $T(PSL(2, 7))$, A_6 and $T(S6)$ are given as follows.*

$$\dim T(PSL(2, 7)) = 165, \dim T(A_6) = 336, \dim T(S_6) = 758.$$

Proof. We compute a set of linearly independent elements among $E_i^* A_j E_k^*$ and $E_i^* A_j E_k^* \cdot E_k^* A_l E_m^* = E_i^* A_j E_k^* A_l E_m^*$. By direct calculation we can see that any form $E_i^* A_{i_1} E_j^* \cdot E_j^* A_{i_2} E_k^* \cdot E_k^* A_{i_3} E_l^*$ linearly depends on the $E_i^* A_{i_4} E_l^*$'s and the $E_i^* A_{i_5} E_{k_1}^* \cdot E_{k_1}^* A_{i_6} E_l^*$'s. Therefore the products of more than two elements of the form $E_i^* A_j E_k^*$ give no new elements of a basis. $\qquad\square$

[1]

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[1]This answers a question raised by Prof.Terwilliger. Indeed our original argument had a gap. He informed us the reference [4]. Our result $\dim T(S_6)$ = 758 is violated to Conjecture 3.5.

We provide the matrices below to show how many elements of a basis occur. As these matrices are symmetric, we omit the entries below diagonal. These matrices are indexed by the conjugacy classes in the order assumed earlier. The entries of matrices indicate the dimension of each position.

For example, the entry 6 in the (C_2, C_2)-position for the group $PSL(2, 7)$ comes from the dimension of subspace that is the product of entry $E_2^* A_i E_j^*$ and $E_k^* A_l E_2^*$. The dimension coming from $E_2^* A_i E_2^*$ is 5 and the product of $E_2^* A_i E_j^*$ and $E_k^* A_l E_2^*$ has dimension 6.

$$
PSL(2, 7): \quad
\begin{array}{cccccc}
1 & 1 & 1 & 1 & & \\
 & & & 1 & 1 & \\
1 & 13 & 7 & 7 & 5 & 10 \\
6 & & 6 & & 3 & 6 \\
 & & & 6 & & 36 \\
 & & & & 4 & 5 \\
 & & & & & 9
\end{array}
$$

$$
A_6: \quad
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & & \\
 & & & 1 & 1 & & \\
9 & 5 & 5 & 9 & 7 & 7 & \\
 & 8 & 8 & 9 & 8 & 8 & \\
 & & 8 & 9 & 8 & 8 & \\
 & & & 16 & 13 & 13 & \\
 & & & & 12 & 12 & \\
 & & & & & 12 &
\end{array}
$$

$$
S_6: \quad
\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 3 & 4 & 4 & 5 & 3 & 2 & 4 & 2 & 3 \\
 & 6 & 4 & 6 & 9 & 8 & 2 & 6 & 4 & 6 \\
 & & 8 & 8 & 8 & 7 & 4 & 8 & 4 & 8 \\
 & & & 12 & 13 & 13 & 4 & 12 & 6 & 13 \\
 & & & & 19 & 16 & 3 & 13 & 6 & 12 \\
 & & & & & 23 & 3 & 13 & 8 & 16 \\
 & & & & & & 3 & 4 & 3 & 5 \\
 & & & & & & & 12 & 6 & 13 \\
 & & & & & & & & 6 & 9 \\
 & & & & & & & & & 19
\end{array}
$$

We denote by $Z(T(G))$ the center of the Terwilliger algebra $T(G)$ of a finite group G.

Lemma 3.2. *The dimensions of $Z(T(G))$ for $G = PSL(2, 7), A_6, S_6$ are given as follows.*

$$\dim Z(T(PSL(2, 7))) =$$
$$7, \quad \dim Z(T(A_6)) =$$
$$10,$$

$$\dim Z(T(S_6)) = 14.$$

Proof. The result is obtained by determining a basis for the center. We solve a linear equation system $\{x_i y = y x_i\}$ ranging over all the basis of $T(G)$ and $\sum c_j b_j$, where b_j are the basis elements of and c_j is any scalar. □

Let $\{e_i : 1 \le i \le s\}$ be a basis of $Z(T(G))$. Then we have $e_i e_j = \sum_k t^k_{ij} e_k$ and put $B_i := (t^k_{ij})$ for $1 \le i \le s$. Since these matrices mutually commute, they are simultaneously diagonalizable. We shall denote by $v_1(i), \ldots, v_s(i)$ the diagonal entries of the diagonalized matrix of B_i and define the matrix M by $M_{ij} := v_i(j)$. Then we get the primitive central idempotents $\varepsilon_1, \ldots, \varepsilon_s$ by

$$(\varepsilon_1, \ldots, \varepsilon_s) = (e_1, \ldots, e_s) M^{-1}.$$

Theorem 3.3. *The degrees of the irreducible complex representations af- forded by every idempotent are given below.*

$T(PSL(2, 7))$

ε_i	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7
$\deg \varepsilon_i$	1	2	3	3	5	6	9

$T(A_6)$

ε_i	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8
$\deg$	1	3	3	4	4	6	6	7
ε_i	ε_9	ε_{10}						
$\deg$	8	10						

$T(S_6)$

ε_i	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8
$\deg \varepsilon_i$	1	1	1	3	3	4	6	7
ε_i	ε_9	ε_{10}	ε_{11}	ε_{12}	ε_{13}	ε_{14}		
	8	89	9		11	15		

Proof. This is because that $T(G)\varepsilon_i \cong M_{d_i}$ and that $d_i = \dim T(G)\varepsilon_i$ equals the number of linearly independent elements in the set $\{x_j \varepsilon_i\}$, where x_j are the basis elements of T. □

Theorems 3.1 and 3.3 are combined as $165 = 1^2 + 2^2 + 3^2 + 3^2 + 5^2 + 6^2 + 9^2$,

$336 = 1^2 + 3^2 + 3^2 + 4^2 + 4^2 + 6^2 + 6^2 + 7^2 + 8^2 + 10^2$,
$758 = 1^2 + 1^2 + 1^2 + 3^2 + 3^2 + 4^2 + 6^2 + 7^2 + 8^2 + 8^2 + 9^2 + 9^2 + 11^2 + 15^2$.

The degrees of irreducible complex representations afforded by every primitive central idempotents enable us to get the following structure theorem.

Corollary 3.4. *We have that*

$$T(PSL(2, 7)) \cong M_1 \oplus M_2 \oplus M_3 \oplus M_3 \oplus M_5 \oplus M_6 \oplus M_9,$$

$$T(A_6) \cong M_1 \oplus M_3 \oplus M_3 \oplus M_4 \oplus M_4 \oplus M_6 \oplus M_6 \oplus M_7$$
$$\oplus M_8$$
$$\oplus M_{10},$$
$$T(S_6) \cong M_1 \oplus M_1 \oplus M_1 \oplus M_3 \oplus M_3 \oplus M_4 \oplus M_6 \oplus M_7$$
$$\oplus M_8$$
$$\oplus M_8 \oplus M_9 \oplus M_9 \oplus M_{11} \oplus M_{15}.$$

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